Optimum Block Length for Data Transmission with Error Checking

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Synopsis: The efficiency of a block-checking system of data transmission, in which an answer-back block-checking signal is provided to acknowledge each block of characters, or to ask for a repeat if an error is detected, is investigated. Curves of data transmission efficiency, as a function of the number of characters in a message block, are shown to have sharper peaks as the distance and error probabilities are increased. For comparison, efficiency curves are plotted for a 7-bit single-error-detecting code and for a 10-bit single-error-correcting code. A family of curves of optimum block length is given for different probabilities of error and different reply times.

A BLOCK-CHECKING SYSTEM is one in which there is a periodic feedback signal which acknowledges correct receipt of the last block of n characters or asks for a repeat of the last block, as illustrated in Fig. 1. A block-check symbol is added after the nth character of the message block. It may be a simple end-of-block signal, or it may contain a parity count of the bits in particular rows of the message. To distinguish between different types of parity checks, a message can be visualized as a matrix where each column contains the bits of a single character. Then the addition of a parity bit to each character constitutes a vertical check. The block-check symbol then becomes a longitudinal check which can consist of the parity bits for each row. A special case of a block-check symbol providing a longitudinal check on the vertical check symbols is described by Barbeau.[1]

Two definitions of optimum block length have come to the attention of the author. Schatzoff and Harding proposed an investigation of the optimum block length for minimizing the undetected errors.[2] A. Cohen, in an unpublished report on transmission of punched card data over teleprinter lines, defined optimum block length as the number of characters per block which would maximize the transmission efficiency. Cohen's definition of optimum block length is used in this study.

The transmission efficiency is the time required to transmit the information bits divided by the total time required to transmit the information bits plus the time required for redundancy bits, waiting time for the answer-back signals, and the average additional time for repeated transmissions.

Bishop and Buchanan have proved a theorem that redundancy for error correction is more economical than using feedback to request a repeat when an error occurs.[3] However, automatic transmission from a computer to a remote station requires some feedback signal to verify that the channel is satisfactory. This problem differs from the conditions of Bishop and Buchanan in that both feedback and redundaney are required

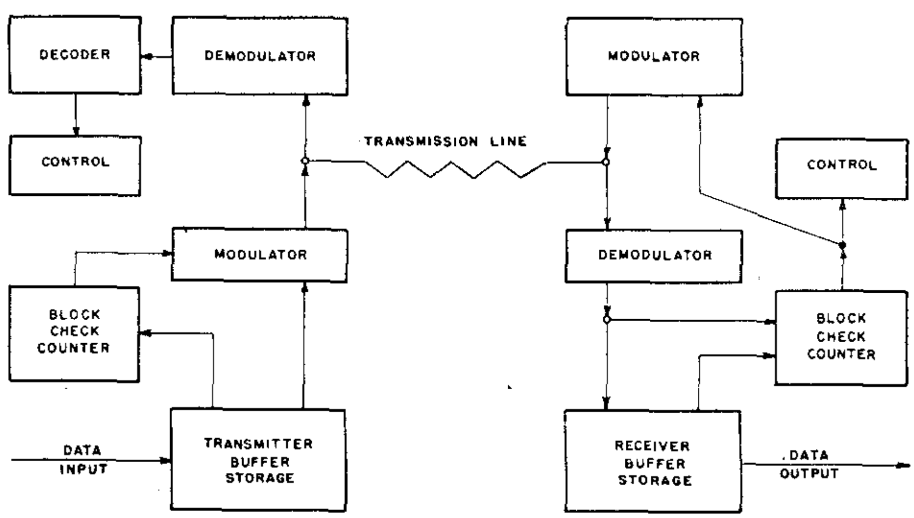

Fig. 1. Block-check answer-back system

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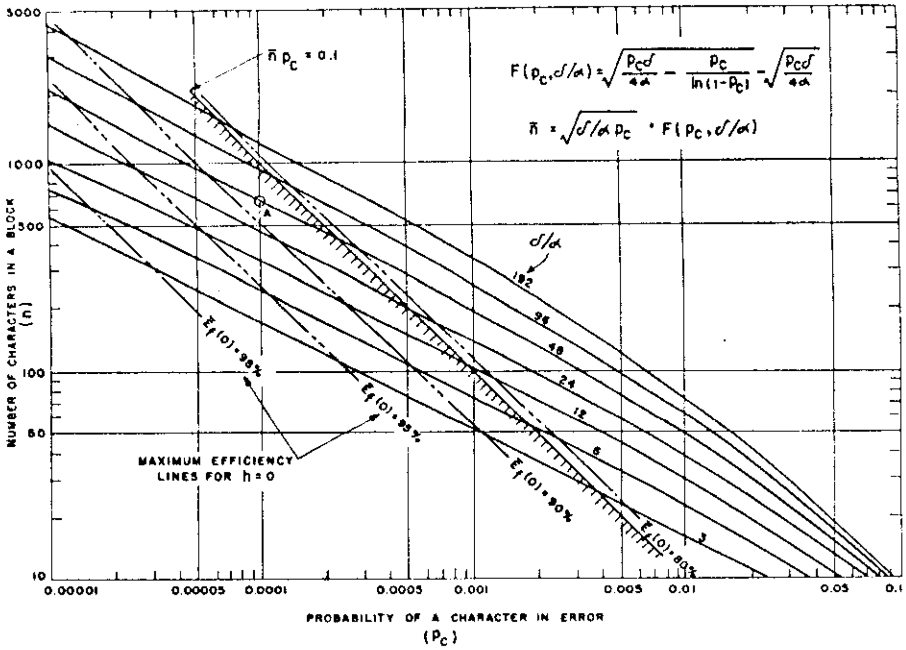

Fig. 2. Optimum block length versus probability that a character is in error; loci of maximum efficiency

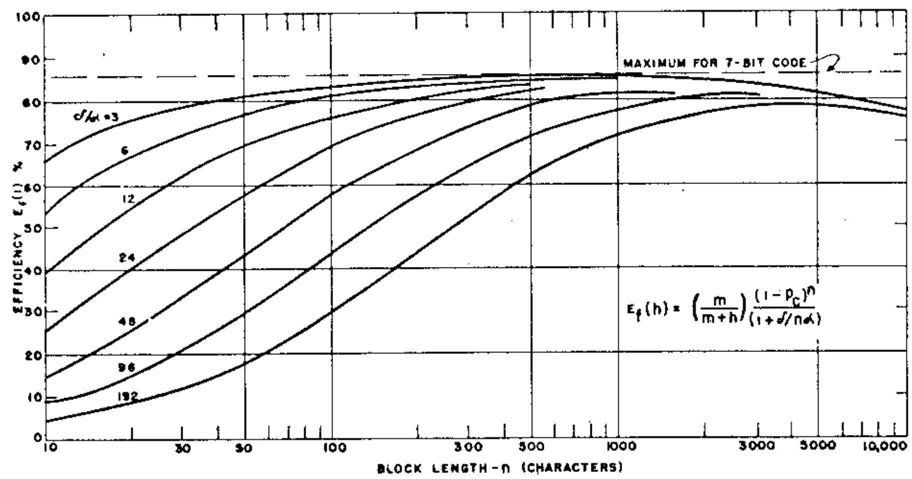

Fig. 3. Data transmission efficiency for specified error reply time δ/α and probability of character error p_e as function of block length n

$h=1$, $p_e=0.00001$

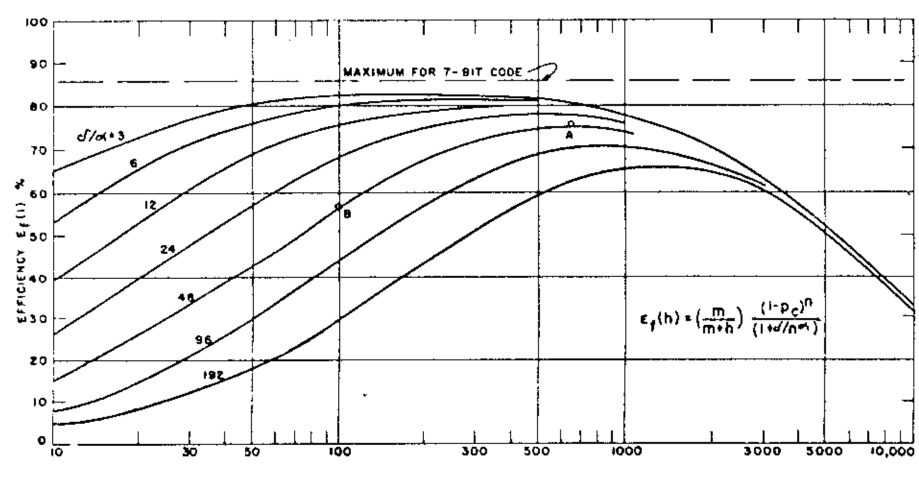

Fig. 4. Same as Fig. 3

$h=1$, $p_e=0.0001$

so that the question is to determine the most economical ratio between redundancy and feedback for a particular set of conditions.

The effect of line failure when the automatic switching system fails to find a spare channel is not included in this analysis. For data on the probability of such events, the reader is referred to reports of Bell Telephone Laboratories.[4]

An answer-back signal is a signal sent by the receiving station to indicate acknowledgment or to request a repeat of the last block. When no errors are detected, the net information transmission rate is reduced by the time involved in coding the block-check symbol, propagation time over long line, decoding logic time, and setting of telephone line echo suppressors. When errors are detected, the net transmission rate is further reduced by the time required to repeat blocks containing errors.

Efficiency is plotted for some sample conditions of error probability and total block-check and answer-back time. A general equation for optimum block length is derived by setting the derivative of the efficiency with respect to the number of characters in a block equal to zero to obtain the condition of maximum efficiency. A set of curves is plotted for different values of reply time and character error probability.

The general equation for optimum block length reduces to Cohen's equation when the product of the block length and the probability of error are small. Sample values of optimum block length and efficiency are plotted for a range of error probabilities and transmission speeds.

The curves in this report can be used to determine the optimum block length and efficiency in practical cases by the following steps:

1. Calculate time interval per character.
2. Calculate the reply delay time interval required for the block-check and answer-back signals, and determine the number of characters which could be sent in the reply delay interval.
3. Select the curve in Fig. 2 corresponding to the number of characters determined in item 2, then determine from statistical data the probability that a character is in error. The intersection of the character error probability with the optimum block length curve corresponding to the number of characters per delay time interval gives a point which graphically yields two items:

   a. The ordinate gives the optimum block length in the number of characters per block.

   b. A projection from the point parallel to the diagonal lines shows the maximum efficiency that can be obtained. These diagonal lines are computed for zero redundancy, so they must be multiplied by the ratio of information to total bits per character.

4. To determine the efficiency at a block length other than the optimum value use equation 13 or read the value off the efficiency curves of Figs. 3 through 6.

A sample calculation is included in Appendix III for the foregoing steps. The use of these curves permits a comparison of line rental costs with buffer storage costs. It also is of use for comparison with error correction codes. Curves for a 10-bit single-error-correcting code are included in Fig. 7 for comparison with the more frequently used 7-bit single error-detecting codes.

The derivation of the efficiency exhibits an interesting phenomenon of different probability functions for the two levels of treatment of the problem. The probability of different numbers of characters being in error within a block follows the binomial distribution. When the probability of a block of n characters being correctly transmitted is considered, the probability follows the Pascal distribution.

Efficiency of Transmission for Single Error-Detecting Code

Definitions

The efficiency of transmission E_f is defined as the time required to transmit the information bits (excluding redundancy bits) of a block of n characters without any error-detecting or error-correcting system, divided by the average time required to transmit a block of n characters with all detected errors corrected. Where T is the average time to transmit a block of n characters with all single and multicharacter errors (detectable by the code used) corrected, T_1 is the time to transmit a block without correcting the errors, and each character contains m information bits and h redundant bits, the foregoing definition results in the following:

$$E_f(h)=\left(\frac{T_1}{T}\right)\left(\frac{m}{m+h}\right)\times 100 \quad (1)$$

Where α is the time to transmit one character, and δ is the time between the end of the last character in a record and the beginning of the first character in the next record or the next repeat transmission

$$T_1=n\alpha \quad (2)$$

and a single block with no detected errors, but with a feedback acknowledgment signal, takes a time interval

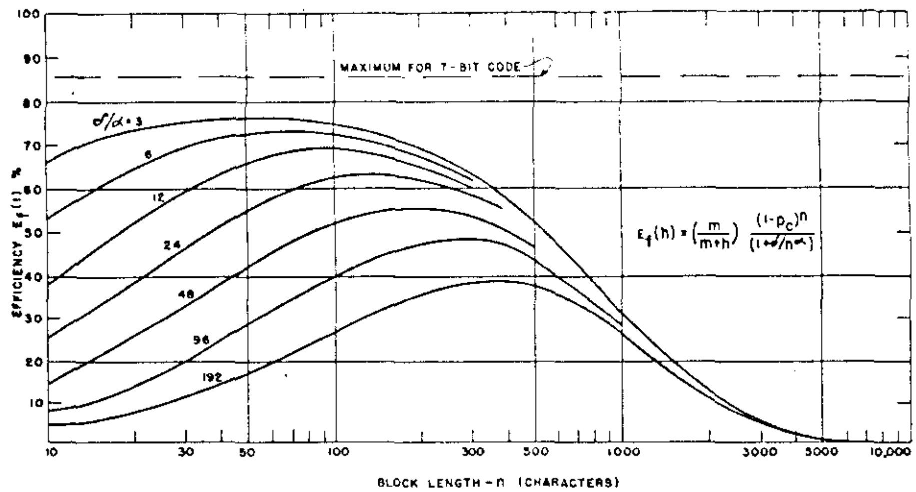

Fig. 5. Same as Fig. 3

$h=1$, $p_o=0.001$

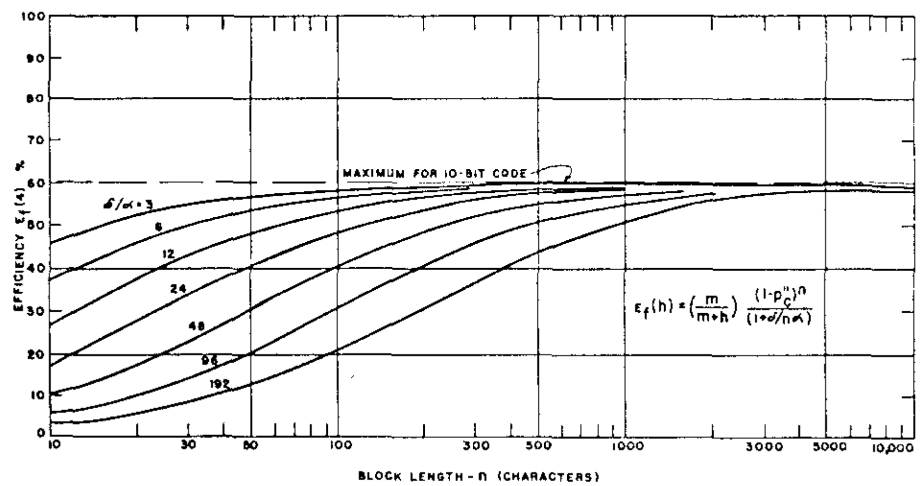

Fig. 6. Same as Fig. 3

$h=4$, $p_o=0.001$; $p_o''\sim p_o^2=0.000001$

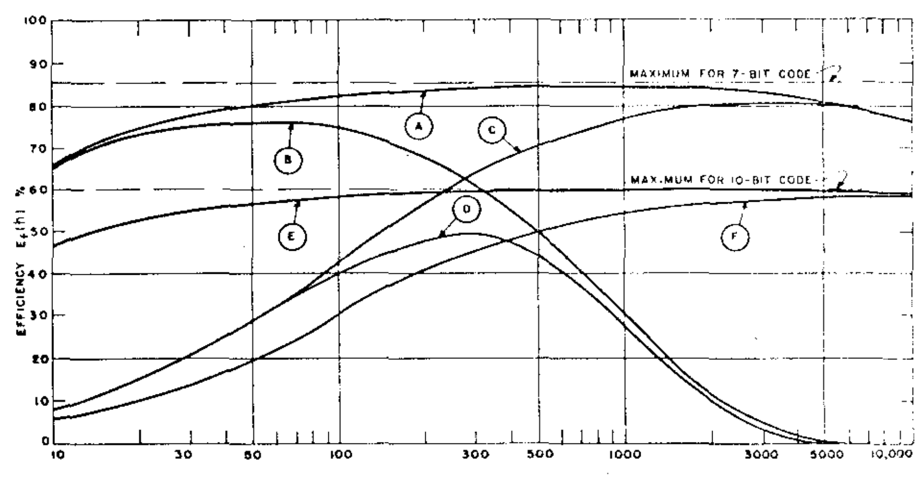

Fig. 7. Same as Fig. 3

A—7-bit code, $\delta/\alpha=3$; $p_o=0.00001$
B—7-bit code, $\delta/\alpha=3$; $p_o=0.001$
C—7-bit code, $\delta/\alpha=96$; $p_o=0.00001$
D—7-bit code, $\delta/\alpha=96$; $p_o=0.001$
E—10-bit code, $\delta/\alpha=3$; $p_o=0.001$; $p_o''\sim 0.000001$
F—10-bit code, $\delta/\alpha=96$; $p_o=0.001$; $p_o''\sim 0.000001$

$$T_0 = n\alpha + \delta \tag{3}$$

The average time to transmit a message block is predicted theoretically by a series of Bernoulli trials where X is the number of times a message must be sent to get a correct transmission and $E(X)$ is the expectation or average value of X, so that

$$T = T_0 E(X) \tag{4}$$

The expectation $E(X)$ is defined mathematically in Appendix I and the equation is derived there for this particular problem.

Average Number of Transmissions per Block Due to Independent Statistical Character Errors

The symbols p_c and q_c are the probability of failure and the probability of success, respectively, of a character being transmitted correctly, where

$$q_c = 1 - p_c \tag{5}$$

It is assumed that the probability of a character being in error is based upon statistically independent events. The series of Bernoulli trials discussed in Appendix I gives the following equation for the expectation

$$E(X) = q_c^n \sum_{k=1}^{\infty} k(1-q_c^n)^{k-1} \tag{6}$$

Changing to blocks of characters, the symbols p_s and q_s are the probability of success and the probability of failure respectively, in transmitting a correct message of n characters. The relationship between the two probabilities is

$$p_s = q_c^n = (1-p_c)^n \tag{7}$$

$$q_s = 1 - p_s = (1-q_c)^n \tag{8}$$

It should be noted that p_s represents success, while p_c represents failure. This change in the role of the symbol p, when changing from characters to blocks of characters, causes the resultant equations for expectation to be in a standard form. Substituting q_c from equations 7 and 8 into equation 6 transforms the expectation into the following form:

$$E(X) = p_s \sum_{k=1}^{\infty} k q_s^{k-1} \tag{9}$$

Comparing equation 9 with the defining equation 19 in Appendix I shows that the probability function is of the form

$$P(X=k) = q_s^{k-1} p_s \tag{10}$$

which is recognized as the Pascal distribution described by Feller.[5] The terms under the summation sign of equation 9 have the form of a binomial series

$$\sum_{k=1}^{\infty} k q_s^{k-1} = (1-q_s)^{-2} \tag{11}$$

Substituting equation 11 into equation 9 results in the following value of the expectation

$$E(X) = \frac{p_s}{(1-q_s)^2} = \frac{1}{p_s} = \frac{1}{(1-p_c)^n} \tag{12}$$

The change in units used from single characters to blocks of characters results in a change from binomial distribution to the Pascal probability distribution.

Efficiency

Using equations 1, 2, 3, 4, and 12 gives a transmission efficiency of

$$E_f(h) = \left(\frac{m}{m+h}\right) \frac{(1-p_c)^n}{(1+\delta/n\alpha)} \times 100 \tag{13}$$

Sample curves are plotted in Fig. 7 for $p_c = 0.001$ and $p_c = 0.00001$. The curves marked $h=1$ are for the 7-bit single-bit error-detecting code with assumption that higher order bit errors are detected by a longitudinal block check plus interlacing of characters if required to separate multiple-bit errors. Two sets of curves are shown, one for an answer-back time of three character time intervals representative of short lines, and a set for 96-character intervals, representative of a 3,000-mile nonloaded cable with echo suppressors at 2,000 bps (bits per second).

Additional curves for a range of δ/α for $p_c = 0.001$, 0.0001, and 0.00001 are plotted in Figs. 3 through 5 so that the efficiency can be interpolated from the curves in this range.

Optimum Block Length for Single-Error Detection

To obtain an equation for optimum block length, the derivative of the efficiency is taken with respect to n and set equal to zero, which gives the maxima and minima of the efficiency curves. Then, choice of the correct root corresponding to a maximum gives the optimum block length. The details of the development are given in Appendix II. Taking the positive root of equation 25 gives the optimum block length

$$\bar{n} = \sqrt{\delta/\alpha p_c}\, F(p_c, \delta/\alpha) \tag{14}$$

$$F(p_c, \delta/\alpha) = \sqrt{[1/S(p_c)] + (\delta/\alpha)p_c/4} - \sqrt{(\delta/\alpha)p_c/4} \tag{15}$$

When $p_c < 1/(100\,\delta/\alpha)$, which may also be stated as $\bar{n}p_c < 0.1$, $F(p_c, \delta/\alpha)$ approaches unity, so that the following equation has less than 5% error:

$$\bar{n}' = \sqrt{\delta/\alpha p_c} \tag{16}$$

Curves of optimum block length are plotted in Fig. 2 for different reply times (δ/α). The complete formula of equation 14 must be used for $\bar{n}p_c > 0.1$. Equation 16 can be used for $\bar{n}p_c < 0.1$.

Lines of equal efficiency have been added to Fig. 2. Substituting $\bar{n}$ in equation 13 and expanding the $(1-p_c)^n$ term in equation 13 by the binomial expansion, and dropping second-order and higher terms of p_c gives the following equation for maximum efficiency with less than 0.5% error when $\bar{n}p_c < 0.1$:

$$\bar{E}_f(h=0) = \frac{\bar{n} - \delta/\alpha}{\bar{n} + \delta/\alpha} \tag{17}$$

Efficiency for Single-Error Correction Code

For comparison of single-error detection codes ($m=6$, $h=1$) with an answer-back signal with a single-error correction ($m=6$, $h=4$) with answer-back signal, the efficiency of the error-correcting code is needed. In this sense "single error" means a single-bit error within a character. Therefore no repeats will be required for the single errors, but repeats will be asked for in the case of multiple errors detected by the addition of longitudinal block checking. Substituting the p_c'' derived in Appendix III for the 10-bit single-error correcting code of the Hamming[6] type into equation 13 gives

$$E_f(h=4) = \frac{m}{m+h} \times \frac{(1 - 45p_b^2(1-p_b)^8 - \ldots)^n}{(1+\delta/n\alpha)} \simeq \frac{m}{m+h}\,\frac{(1-p_c^2)^n}{(1+\delta/n\alpha)}, \text{ for } p_c \leq 0.001 \tag{18}$$

Examples of Optimum Block Length and Efficiency

Examples have been taken graphically from Fig. 7 to obtain the optimum block length and efficiency for a range of bit rates between 75 bps and 6,000 bps. These examples were based on nonloaded side circuits on a 3,000-mile 19-gage cable. These results are plotted (see Fig. 8) to show the change in optimum block length and maximum efficiency.

Curves have been plotted for lines with and without echo suppressors. Echo suppressors are used on most voice

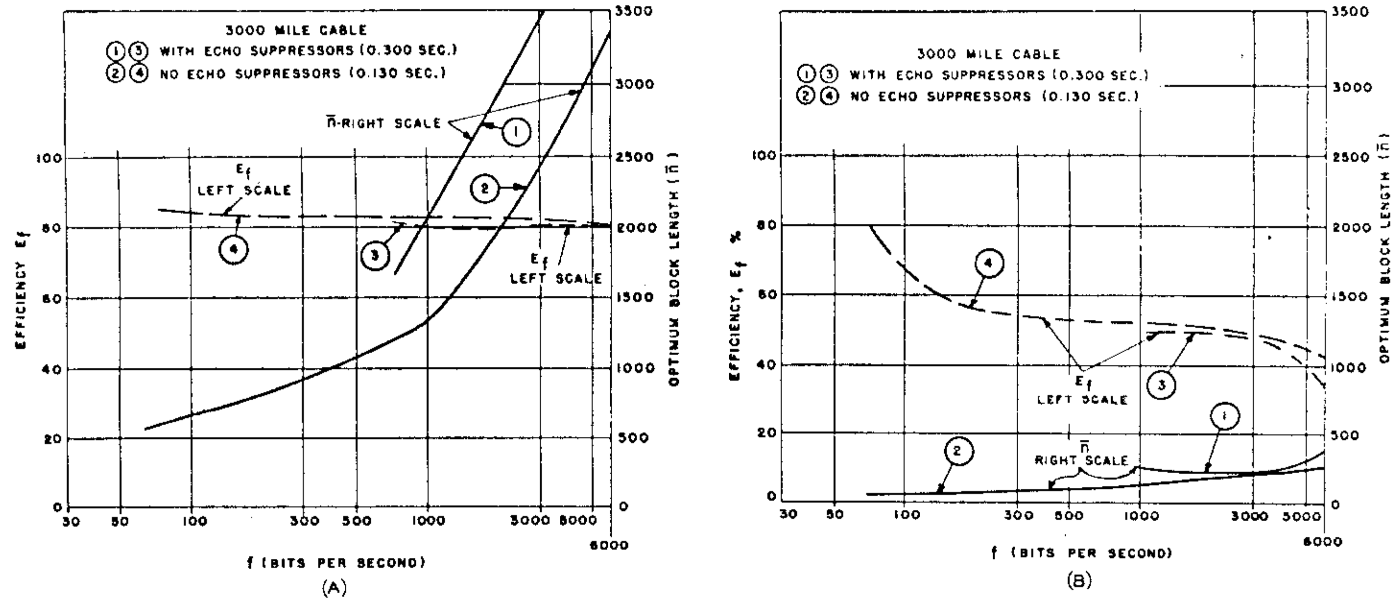

Fig. 8. Variation of optimum block length and efficiency with transmission rates:

A—h=1; p_c=0.00001 B—h=1; p_c=0.001

telephone circuits of over 500 miles. It is common to remove the echo suppressors from leased lines used solely for data transmission. A brief description of the functioning of echo suppressors may be found in a text issued by American Telephone and Telegraph Company.[7] Examination of Fig. 8 shows that the presence of echo suppressors only reduces the maximum attainable efficiency a few per cent, but their presence raises the optimum block length by as much as 50% for low-error probabilities such as $p_c=0.00001$.

This leads to the consideration of the case of a fixed buffer of less than the optimum block length. Some sample values of n and E_f are listed in Table I. If $p_c=0.001$ and $\delta/\alpha=24$, the optimum block length of $\bar{n}=130$ would give an efficiency of 69%, while the use of a buffer holding 100 characters would yield an efficiency of 62.5%. If the error probability decreases, the maximum efficiency goes up to 83% at the optimum block length of 1,300 characters. However, if the original block length of 100 characters is retained, there is still a gain of almost 7% in the efficiency without changing the buffer size to the optimum block length. To decide when it is practical to adjust the buffer size to the optimum requires knowledge of buffer costs and line rental costs to determine whether the buffer costs offset the line rental savings as a result of increased transmission efficiency.

Conclusions

The importance of the optimum-block-length concept increases with an increase in any of the following parameters: 1. data transmission rate, 2. distance of transmission, and 3. probability of error. When these parameters are all low, as in the top curve ($\delta/\alpha=3$, $p_c=0.00001$) in Fig. 7, the efficiency drops only a few per cent over a range of 50 to 5,000 characters per block. As any one of these parameters increases, the curve of efficiency versus block length sharpens.

For a long-distance transmission, the increase in optimum block length indicates the need for larger buffer storage. The efficiency equation can be used in balancing the cost of the larger buffer required for use of optimum block length against possible reduced line rentals due to a higher effective data transmission rate. The calculation of optimum block length and efficiency permits a comparison of the added rental cost of the local loops to obtain complete 4-wire service with the savings from reduced size of buffer storage.

In a system where automatic communication between a computer and remote stations requires a periodic feedback signal to protect against unusual events (such as an increase in multiple errors or line failure when there are no spare channels available to the automatic switching system), the details of the situation must be examined to determine whether feedback correction of errors or error-correction by redundancy is more efficient. For some combinations of delay time (δ/α) and probability of a character being in error (p_c), the 7-bit code is more efficient, while for other combinations the 10-bit code is more efficient.

Table I. Operation at Nonoptimum Block Lengths

f, Bps	p_c	δ/α	Optimum $\bar{n}$	Optimum E_{max}, Per Cent	Sample Nonoptimum n_{no}	Sample Nonoptimum E_{no}, Per Cent
1,000	0.00001	24	1,300	83	100	69.1
1,000	0.0001	24	460	77.5	100	68.4
1,000	0.001	24	130	63	100	62.5

Nomenclature

a = number of bits per character
$E(X)$ = expectation of event X or the average value of X
$E_f(h)$ = transmission efficiency with h redundant bits per character
$\bar{E}_f(h)$ = maximum transmission efficiency obtainable when optimum block length is used
f = transmission rate in bps
$F(p_c,\delta/\alpha)$ = correction factor to obtain exact optimum block length from the approximate equation
h = number of redundant bits per character
i = summation index indicating number of bits or characters in error

k = summation index indicating number of blocks
l = length of transmission channel
m = number of information bits per character
$\bar{n}$ = number of characters per block
n = optimum number of characters per block
$p(x_k)$ = probability of occurrence of event x on the kth trial
p_b = probability that a bit is in error
p_c = probability that a characteris in error
p_c'' = probability that a character is in error due to two or more bits in error within one character
$p_f(k)$ = probability that a block of characters has to be repeated k times to obtain a correct transmission
p_s = probability that a message is correct on the first transmission
$P(X)$ = probability of set of events defined by X occurring
q_c = probability that a character is correct
q_s = probability that a message is not correctly received on one transmission
$S(p_c)$ = a factor in the optimum block length equation which is asymptotic to unity for small values of P_c
T = average time to transmit a block of n characters with all single and multi-character errors corrected
T_0 = time taken to transmit a block of characters with no errors, but with reply time added to acknowledge message
T_1 = time taken to transmit a block of characters without any feedback signal
X = a set of events such as i characters in error
x_k = an event which occurs on the kth trial
α = time interval of one character
δ = reply delay time interval
δ/α = number of characters which could be transmitted during the reply delay time interval

Appendix I. Derivation of Average Number of Transmissions per Block

To obtain the average time per message T from equation 4, a formula for the expectation is derived from the following definition given by Feller.[8]

Let x be a random variable assuming value x_1, x_2, ... with corresponding probabilities $p(x_1)p(x_2)$. The mean or expected value of x is defined by

$$E(x) = \sum_{k=1}^{\infty} x_k p(x_k) \qquad (19)$$

provided that the sum converges absolutely.

The grouping in blocks of n characters sets integral values on x_k, so that $x_k = k = 1, 2, 3, \ldots$ When the probability of a character being in error is independent, the binomial probability distribution applies, so

$$P\,(i \text{ characters in error}) = \frac{n!}{i!(n-i)!}\, p_c^i q_c^{n-i} \qquad (20)$$

These repeated independent trials constitute a series of Bernoulli trials as defined by Feller.[9]

Defining $p_f(k)$ as the probability that the block has to be transmitted k times, to obtain a correct transmission on the kth trial yields

$$p_f(1) = q_c^n = (1-p_c)^n$$

$$p_f(2) = \left[\sum_{i=1}^{n}\binom{n}{i} p_c^i q_c^{n-i}\right] q_c^n = (1-q_c^n)\, q_c^n$$

$$p_f(3) = (1-q_c^n)^2 q_c^n \qquad (21)$$

Extending equations 21 to the general form and substituting it into equation 19 gives the expectation

$$E(X) = q_c^n \sum_{k=1}^{\infty} k(1-q_c^n)^{k-1} \qquad (22)$$

The foregoing gives the average transmission time when substituted into equation 4.

Appendix II. Derivation of Optimum Block Length

Taking the derivative of equation 13 and setting equal to zero gives

$$\frac{dE}{dn} = \left(1+\frac{\delta/\alpha}{n}\right)^{-1} \frac{d}{dn}(1-p_c)^n + (1-p_c)^n \times \left(1+\frac{\delta/\alpha}{n}\right)^{-2}\left(\frac{\delta}{\alpha}\right) n^{-2} = 0 \qquad (23)$$

$$\frac{d}{dn}(1-p_c)^n = (1-p_c)^n \ln(1-p_c)$$

Let

$$S(p_c) = [-\ln(1-p_c)]/p_c = 1 + \frac{p_c}{2} + \frac{p_c^2}{3} + \ldots + \frac{p_c^{x-1}}{x} + \ldots \qquad (24)$$

$$(\bar{n})^2 + (\bar{n})(\delta/\alpha) - [(\delta/\alpha) \quad /\ln(1-p_c)] = 0$$

$$\bar{n} = \frac{-(\delta/\alpha) \pm \sqrt{(\delta/\alpha)^2 + 4\delta/\alpha p_c S(p_c)}}{2} \qquad (25)$$

Taking the positive root in equation 25 gives the optimum block length

$$\bar{n} = \sqrt{\delta/\alpha p_c}\; F(p_c, \delta/\alpha) \qquad (14)$$

where $F(p_c, \delta/\alpha)$ is given by equation 15.

Appendix III. Sample Calculation and Graphical Determination of $\bar{n}$ and E

Conversion of Bit Error Probability to Character Error Probability

When p_b is the probability that a bit is in error in a binary symmetric channel where bit errors are independent, and it is assumed that a block check character provides a longitudinal check on multiple bit errors, the probability of a character being in error (and being detected at the receiver) is

$$p_c = \sum_{i=1}^{7}\binom{7}{i} p_b^i (1-p_b)^{7-i} \qquad (26)$$

Equation 26 applies for $m=6$, $h=1$ (single-error-detection code, double and higher bit errors to be detected by longitudinal block check).

Sample calculation for $p_b = 0.00014$ is as follows:

$$p_c = 7p_b(1-p_b)^6 + 21p_b^2(1-p_b)^5 + \ldots$$
$$= 7(0.00014)(0.99986)^6 + 21(0.00014)^2 \times (0.99986)^5 + \ldots$$
$$= 0.000\,98 + 0.000\,00041 + \ldots \sim 10^{-3}$$

For $m=6$, $h=4$ (single-error correction, double and higher order bit error to be detected by longitudinal block check), the corresponding formula is

$$p_c'' = \sum_{i=2}^{10}\binom{10}{i} p_b^i (1-p_b^i)^{10-i}$$

$$= p_c'' = 45p_b^2(1-p_b)^8 + 120p_b^3(1-p_b)^7 + \ldots$$
$$\simeq 0.000\,000\,9 + 0.000\,000\,000\,336 +$$
$$\simeq 0.000\,000\,9 \simeq 10^{-6} \qquad (27)$$

$$p_c'' \sim p_c^2, \text{ for } p_c < 0.001 \qquad (28)$$

Graphical Determination of $\bar{n}$ and E

Condition serial transmission, 1,000 bps, $p_c = 0.0001$, $l = 3{,}000$ miles.

Step 1:

$$\alpha = \frac{a}{f} = \frac{7 \text{ bits per character}}{1{,}000 \text{ bps}} = 0.007 \text{ second per character}$$

Step 2: Take $l = 3{,}000$ miles, assume typical transmission line with a total of 0.300 (second) delay time for 2-way propagation and resetting of the echo suppressors Assume that two characters are used for block-check and answer-back signals and that the logical operations can be completed within one character time, so that $\delta/\alpha = 3 + (0.300/0.007) = 47$.

Step 3: In Fig. 2 the intersection of the curve $\delta/\alpha = 47$ with $p_c = 0.0001$ is marked as point A. This gives optimum block length $\bar{n} = 640$, and maximum efficiency $\bar{E} = (6/7)86 = 74\%$.

Step 4: To determine the efficiency when 100 characters are used as the block length so that $n = 100$ characters (see Fig. 4), the exact result can be obtained from equation 13 or the efficiency can be obtained by interpolating between the curves of Fig. 4. Point A for $\bar{n} = 640$ and point B for $n = 100$ are shown in Fig. 4. From point B it is determined the the efficiency is reduced to 57% when a block length of 100 characters is used.

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