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Coupling between Waveguides and Cavity Resonators for Large Power Output

By

Frederick Bernard Wood
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I. Introduction

1.1 Cavity Resonator Coupling Devices

The four basic types of coupling to cavity resonators are loop, probe, iris, and electron beam couplings which are illustrated in figure 1.1. Coupling by loops was first described by Hansen,[1] who also noted that capacitive coupling was possible. Hansen also used Rayleigh's[2] results on the diffraction of a circular aperture to compute the radiation losses due to holes in cavity resonators. Bunimovich[3] treated probe and loop coupling from a two wire line to a cavity resonator. Neiman[4] obtained the electric and magnetic dipole moments for small circular and rectangular irises in cavity resonators. Condon[5] developed a more general theory of loop and probe coupling. Bethe independently obtained a solution for the diffraction of a small hole in an infinite screen[6] and extended the results to the coupling of cavity resonators by small irises.[7] Schelkunoff[8] developed a more general representation of impedance functions for transmission lines and resonators, and treated an example of a probe linking the full height of a resonator. Wartime developments in coupling to cavity resonators are summarized by the M.I.T. Radiation

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1. Hansen, W.W., Jour. Appl. Phys., Vol 9, Oct. 1938. pp 654-663.
2. Rayleigh, Philosophical Mag., Series 5, Vol 44, 1897. pp 28-52.
3. Bunimovich, V.I. Jour. of Tech. Phys (U.S.S.R.), Vol 9, No. 11, 1939. pp 984-1004 (In Russian).
4. Neiman, M.S. Izvestija Electropromyshlennosti Slabogo Toka, No. 6, 1940. pp 1-16 (In Russian).
5. Condon, E.U., Jour. Appl. Phys., Vol 12, Feb. 1941. pp 129-132.
6. Bethe, H.A., M.I.T. Rad. Lab. Report 128 (V-15S), Jan. 23, 1942.
7. Bethe, H.A., M.I.T. Rad. Lab. Report 194 (43-22), March 24, 1943.
8. Schelkunoff, S.A., Proc. I.R.E. Vol 32, Feb. 1944. pp 83-90.

Laboratory Series[9] and by the Bell Telephone Laboratories.[10] All four types of couplings are reviewed by Bernier.[11] Lucke has obtained the transmission cross-section for a rectangular aperture in an infinite screen by a variational method.[12] Maltzer[13] has obtained experimental curves of the coupled Q and resonant frequency for one particular resnatron anode resonator. A more complete list of refernces is included in the bibliography at the end of this report.

1.2 The Iris Coupling Problem

The simplest and most practical construction for large power output from a resonant cavity is by iris coupling to waveguide. Bethes' lumped constants apply only to small size irises which can only couple a small amount of power compared to the requirements of high power resnatron[14] oscillator and amplifier tubes. There is another method for coupling large amounts of power from a resonator which is not treated in this analysis, namely a ring iris coupling to a coaxial line whose axis coincides with the resonator axis.

9. Montgomery, et. al. Principles of Microwave Circuits (1948) pp 183-186, 207-239.
10. Bell Telephone Laboratories, Radar Systems and Components (1949) pp 909-1020.
11. Bernier, J., Annales de Radioelectricite, Vol IV, Jan 1949. pp 3-11.
13. Maltzer, I., I.E.R. Report Series I, Issue No. 43, Microwave Laboratory, U. of C., Berkeley. June 1, 1951.

14. Salisbury, W. W. Electronics. Vol. 19 Feb 1946, pp 92-97.

12. Lucke, W.S., Technical Report No. 25, SRI Project No. 188, Stanford Research Institute, Stanford. September 1951.

Since it would be desirable to have a method of approximating the coupled Q for various resonator shapes, it was decided to make this study on a simple resonator which would have a general usefullness, in that the field distribution at the iris is similar to various specific resonators encountered in practice. A $TEM_{o, o, 3}$-mode coaxial resonator coupled by a rectangular iris to a $TE_{1,0}$- mode rectangular waveguide is investigated for the coupled Q and resonant frequency as function of the iris dimensions. Figure 2,1 is a sketch of the experimental resonator. The dimensions were chosen to be a compromise between a shape corresponding to several resnatron anode resonator problems and a shape that would give a standing wave ratio at resonance which could be experimentally measured by standard techniques.

This study is restricted to a single iris coupling a waveguide to a cavity resonator. The change in coupled Q and resonant frequency is investigated theoretically and experimentally. When the desired coupled Q is calculated from a knowledge of the electron beam input to a cavity resonator, then the proper iris dimensions to obtain the desired Q can be determined from the results of this analysis, subject to the error introduced by neglecting possible higher mode coupling between electron beam and the output iris.

The study of coupling of large iris can logically proceed from the existing theory for small irises in four steps:

(a) The first question considered is what is the practical limit to the use of Bethes' approximations in engineering practice.

(b) The next question is to develop a better approximation for medium size irises. The fact that Bethes' approximation uses the results for a small iris in an infinite plane as an approximation to the coupling for a small iris in a cavity resonator leads one to estimate that the polarizability of a transverse iris in an infinite waveguide substituted into the equations of Bethe for using lumped constants might be a better approximation to the coupling between a waveguide and a resonator.

(c) Then the solution of the coupling problem for a simple rectangular resonator when recast into lumped constant form offers a still better approximation for medium irises.

(d) Then for very large irises the problem is to apply the mathematical techniques of electromagnetic diffraction theory directly to the particular iris coupling between a waveguide and cavity resonator.

1.3 Theoretical Techniques from Network and Diffraction Theory.

Add Nal

H. A. Bethe[7] obtained approximate formulas for the equivalent electric and magnetic dipole moments per unit field by use of the approximation $ka << 1$, where $k = 2\pi/\lambda$ and a is the distance from the center to the farthest point on the edge of the iris. Bethes' results for the rectangular inductive iris shown in figure 1.2, transposed into M.K.S. units are:

$$P/\epsilon = M_t/\mu = (\pi/16)\, c^2 \ell \qquad (1.1)$$

where P and M_t are the electric and transverse magnetic polarizabilities respectively. He obtains a change of resonant frequency of:

$$\frac{\Delta\omega}{\omega} = \frac{1/2\left[P\, E_{on}E_{2n} + jM_1\, H_{o\ell}H'_{2\ell} + jM_2\, H_{om}H'_{2m}\right]}{1/2\int\left[\epsilon E_2^2 + \mu H_2'^2\right]\, dv} \qquad (1.2)$$

where E_o and H_o are the incident fields, and E_2 and H_2 are the fields exiated on the other side of the iris. The coupled Q is:

$$Q_c = \frac{\omega U_E}{\frac{\omega^2}{8\, S_a}\left| M_1 H_o' H_a + M_2 H'_{om} H_{am} + PE_{on}E_{an}\right|^2} \qquad (1.3)$$

Add more [illegible]

$$S_a = 1/2 \int_{\substack{\text{cross-section}\\ \text{of waveguide}}} (\bar{n} \times \bar{E}_a) \cdot \bar{H}_a \, dS$$

U_e is the peak energy stored in the electric field of the resonator.

Dyadic Green's functions have been applied to waveguide and cavity resonator problems by J.S. Schwinger.[19] He sets up an electric and magnetic dyadic Green's function satisfying a differential equation derived from Maxwell's equations and having the boundary conditions: $n \times \overleftrightarrow{\Gamma}^{(1)}(r,s) = 0$ and $n \times \text{curl}\, \overleftrightarrow{\Gamma}^{(2)}(r,s) = 0$, respectively on the bounding surface S. For a source free region the electric and magnetic fields can be derived from the tangential electric or magnetic field on the boundary; WHERE $\bar{n}$ is the unit normal into the region;

$$\bar{E}(s) = \int_S \left| \bar{n}\times\bar{H}(r) \right| \cdot \Gamma^{(1)}(r,s)\, dS(r) \qquad (1.4)$$

$$\bar{H}(s) = \int_S \bar{E}(r)\times\bar{n} \cdot \overleftrightarrow{\Gamma}^{(2)}(r,s)\, dS(r) \qquad (1.5)$$

These dyadic Green's functions can be expanded in the normal modes as follows: (in M.K.S. units).

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19. Schwinger, J.S., M.I.T. Rad. Lab. Report 205 (43-34), May 21, 1943.

$$\overleftrightarrow{\Gamma}^{(1)}(r,s) = + j\omega \left\{ \sum_a \frac{\overline{A}_a(r)\ \overline{A}_a(s)}{\omega_a^2 - \omega^2} - \frac{1}{\omega^2} \sum_i \overline{A}_i(r)\ \overline{A}_i(s) \right\} \quad (1.6)$$

$$\overleftrightarrow{\Gamma}^{(2)}(r,s) = + j\omega \left\{ \sum_a \frac{\overline{F}_a(r)\ \overline{F}_a(s)}{\omega_a^2 - \omega^2} - \frac{1}{\omega^2} \sum_k \overline{F}_k(r)\ \overline{F}_k(s) \right\} \quad (1.7)$$

Schwinger[16] applied a variational principle to the solving of diffraction problems by putting the integral equations (1.4) and (1.5) into a stationary form where the error in the answer is proportional to the square of the error in the approximation used as a trial field. For example, where Y is an unknown admittance, $\overline{h}(r)$ is the dominant mode function, $K_m(r)$ is the unknown field in the aperture, and $\overleftrightarrow{\Gamma}^{(2)}(r,s)$ is the Green's function for the region, Schwinger obtains the following integral equations; where $\overline{n}_r$ is the unit normal into the resonator volume.

$$\overline{H}(s) = I\ \overline{h}(s) = \int_S \overline{K}_m(r) \cdot \overleftrightarrow{\Gamma}^{(2)}(r,s)\ dS(r) \quad (1.8)$$

$$\frac{1}{Y} = \frac{I\ V}{I^2} = \int \overline{n} \cdot (\overline{E} \times \overline{H})\ dS = \frac{\int \overline{K}_m(r) \cdot \overline{h}(r)\ dS(r)}{I} \quad (1.9)$$

$$Y = \frac{I\ V}{V^2} = \frac{\int dS(r) \int dS(s)\ \overline{K}_m(r) \cdot \overleftrightarrow{\Gamma}^{(2)}(r,s) \cdot \overline{K}_m(s)}{\left[\int \overline{K}_m(r) \cdot \overline{h}(r)\ dS(r) \right]^2} \quad (1.10)$$

Equation (1.8) is identically equation (1.5) with $\overline{K}_m = \overline{E} \times \overline{n}$ When Y is a real number, equations (1.9) and (1.10) respectively give a lower and upper bound for Y upon insertion of a trial value for $\overline{K}_m(r)$.

In many diffraction problems such as thin irises in infinite waveguides the imaginary part of $Y = G_1 + jB$ is all that is required for a solution. In cavity resonator coupling problems the wall losses

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16. Saxon, D.S., Notes on Lectures by Julian Schwinger. Unpublished, but widely circulated. Feb. 1945.

make a finite conductance, G, at resonance which requires the removal of the singularities of Green's functions for the theory to correspond to the actual situation. To remove the singularities the method of perturbation of boundary conditions developed by Feshbach[~~15~~ 17] can be used. The boundary condition $\bar{n} \times \operatorname{curl} \bar{H} = 0$ is replaced by:

$$\bar{n} \times \operatorname{curl} \bar{H} = (\bar{n} \times \bar{H}) \cdot \overleftrightarrow{Z} \tag{1.11}$$

where Z is proportional to the wall impedance. Where $\overleftrightarrow{\Gamma}^{(2)}(r,s)$ is the Green's function with infinite conductivity and $\overleftrightarrow{G}^{(2)}(r,s)$ is for finite conductivity:

$$\overleftrightarrow{G}^{(2)}(x,s) = \left\{ \overleftrightarrow{\Gamma}^{(2)}(x,s) - \int \bar{n}_r \times \overleftrightarrow{G}^{(2)}(x,r) \cdot \left[\operatorname{curl}_r \overleftrightarrow{\Gamma}^{(2)}(r,s) + \overleftrightarrow{Z} \cdot \overleftrightarrow{\Gamma}^{(2)}(r,s) \right] dS(r) \right\} \tag{1.12}$$

$\overleftrightarrow{\Gamma}^{(2)}(x,r)$ can be substituted in place of $\overleftrightarrow{G}^{(2)}(x,r)$ under the integral sign and then the integration gives a first approximation to $\overleftrightarrow{G}^{(2)}(x,s)$. Repeated substitution of each approximation to obtain the next constitutes an iteration method for removing the singularities.

Marcuvitz[~~16~~ 18] and Oliner[~~17~~ 19] have used the variational method of Schwinger to obtain the values of the equivalent circuit components of irises in infinite waveguides, in T-junctions, and radiating from a waveguide end into free space. A special case of an iris coupled short-circuited waveguide has been described by Robert Beringer.[~~18~~ 20]

17 ~~15~~. Feshbach, H. Phys. Rev. Vol 65, June 1 and 15, 1944. pp. 307-318.

18 ~~16~~. Marcuvitz, N. Microwave Research Institute Report R-193-49, PIB-137. P.I.B., New York, 1949.

19 ~~17~~. Oliner, A.A. <u>Equivalent Circuits for Slots in Rectangular Waveguide</u>. Air Force Cambridge Research Center, August 1951.

20 ~~18~~. Montgomery, op. cit. pp 231-234.

III. Infinite Plane and Infinite Waveguide Approximations.

3.1 Infinite Plane Approximation

Bethes' lumped constants for small irises come from approximate solutions to the diffraction of an aperture in an infinite plane. A further approximation is introduced by using these results in the coupling between a resonator and waveguide. Bethe specified the limitations that :

$$(kc/2) \ll 1 \qquad (3.1) \qquad [illegible] > [illegible] \qquad (3.2).$$

To determine how stringent limitation (3.1) must be, the experimental results on the $TEM_{0,0,3}$ resonator for Q_e and f_0 are compared with Bethe's formulas in figure 3.1 and 3.2.

For the small irises considered by Bethe, the same lumped constants could be used to determine Q_c and f_0. For large irises the Q and resonant frequency characteristics vary differently so it is more convenient to define separate function. For comparison with experimental results and with other analyses, Bethe's results from equations (1.1) - (1.3) are transformed into the following form: (M.K.S. units)

$$q = \left(\frac{\mu}{M}\right)^2 \quad \text{(iris)} \qquad v = \left(\frac{M}{\mu}\right) \quad \text{(iris)}$$

$$N_g = \frac{8\,S_A}{\omega\mu\,|H|^2} \quad \text{(waveguide)} \qquad N_r = \frac{U_E}{\mu\,|H|^2} \quad \text{(resonator)} \qquad (3.3)$$

Putting (3.3) into (1.2) and (1.3) gives:

$$\Delta\omega/\omega = -v\,4N_r \qquad (3.4)$$

$$Q_c = N_g \cdot q \cdot N_r \tag{3.5}$$

For a $TE_{1,0}$ waveguide: $$N_g = \frac{ab\lambda_g}{\pi} \tag{3.6}$$

For a $TEM_{0,0,3}$ resonator with rectangular iris centered at $u = L/3$:

$$N_r = \frac{\pi}{2} r_2^2 \, L \ln r_2/r_1 \tag{3.7}$$

For the resonator of figure 2.1 at $f = 2820$ mc/s:

$$N_g = 123 \times 10^{-6} \ (\text{meter}^3)$$

89.1

$N_r =$ [redacted] $\times 10^{-6}$ (meter^3)

For a rectangular iris, using equations (1.1) and (3.3):

$$v = \frac{\pi a^2 b}{16} \alpha^2 \beta \qquad q = \frac{256}{\pi a^4 b^2 \alpha^4 \beta^2} \tag{3.8}$$

For this particular resonator:

$$v = 34.8 \times 10^6 \alpha^2 \beta \qquad q = \frac{0.826 \times 10^9}{\alpha^4 \beta^2}$$

These functions q and v are plotted in figure 3.1 along with the experimentally observed coupled Q and shift in resonant frequency for the $TEM_{0,0,3}$ resonator. The experimental results have been normalized by use of equations (3.4) and (3.5) for comparison with Bethe's results. This "normalization" puts the experimental results in a convenient form for use as an approximation to other resonators having a similar field near the iris. Thus the experimental curves of q and v can be used to estimate the Q_c and $\Delta\omega/\omega$ of other resonators by use of equations (3.3) - (3.5).

Comparison of the experimental and approximate theory curves shows that equation (3.1) can be more specifically specified by $\alpha = c/a < 0.2$ or $kc/2 < 0.3$ provided equation (3.2) is satisfied when the wall thickness correction is used. ~~be satisfied.~~ For $\beta = 1.0$ and $\alpha = 1.0$ Bethe's approximation again comes close to the experimental value of Q_c. This does not mean that the approximation is valid for large irises, but is more of a coincidence.

Any approximation for use with large irises must account for the resonance phenomena of a thin half wave capacitive slit and equivalent resonant irises as is shown by the minima in the q curves and by the crossing of the v curves.

An improvement of Bethe's approximation would be to convert the transmission cross-sections for a rectangular slot in an infinite plane obtained by Lucke[23] into an equivalent polarizability. This should be useful for larger irises, because it retains the half wave resonance phonomena which is lost in Bethe's approximation.

3.2 Infinite Waveguide Approximations

The next step is to try to extend theoretically Bethe's system of lumped constants for small irises to the case of medium irises. What would appear at a first glance to be a logical way is to use the known susceptance of irises in a transverse diaphram in an infinite waveguide as a closer approximation. However, it will be shown that this doesn't gain much, because it is the behavior of the iris near the shape for iris resonance that is the more important factor for medium and large size irises.

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23 ~~1~~. Lucke, W.S., op. cit. figures 2 and 3.

IV. Dyadic Admittance Functions Including the Resonator Wall Losses.

4.1 Wall Losses in Resonator Problems

The introduction of a normal mode expansion of the Green's function into an integral equation in variational form like equation (1.12) only gives the reactive part of the admittance or impedance. To obtain the real part of the admittance near a particular resonance, the normal mode eigenfunction for that resonance must be introduced in a way that accounts for the resonator wall losses. If the Green's function is modified to include wall losses to give the finite conductance at resonance which corresponds to the physical situation, the singularities of the Green's function for each eigenvalue $\omega = \omega_\alpha$ is removed, which destroys the orthogonality condition of the normal modes.

The perturbation of boundary conditions of Feshbach will be used here. In equation (1.11) the wall impedance dyodic becomes:

$$\overleftrightarrow{Z} = Z_s \overleftrightarrow{\zeta} \tag{4.1}$$

where

$$Z_s = \sqrt{\frac{i\omega\mu}{\sigma}} \tag{4.2}$$

and

$$\overleftrightarrow{\zeta} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{4.3}$$

where σ is the conductivity of the wall.

Using equation (1.12) and replacing the $\overleftrightarrow{G}^{(2)}(x,r)$ under the integral sign by $\overleftrightarrow{\Gamma}^{(2)}(x,r)$, gives the first approximation:

$$\overleftrightarrow{G}^{(2)}(x,s) \simeq \left\{ \begin{array}{l} \overleftrightarrow{\Gamma}^{(2)}(x,s) \\ -\int_s \bar{n}_u \times \overleftrightarrow{\Gamma}^{(2)}(x,u) \cdot \overleftrightarrow{\zeta} \cdot \overleftrightarrow{\Gamma}^{(2)}(u,s)\, Z_s \, dS(u) \end{array} \right\} \tag{4.4}$$

and using equation (1.7) for $\overleftrightarrow{\Gamma}^{(2)}(x,s)$, and dropping Fk's having zero curl, gives:

$$\vec{G}^{(2)}(x,s) = \vec{\Gamma}^{(2)}(x,s) + \sum \sum \frac{\omega^2 F_{\alpha\beta}\, F_{\alpha}(x)\, F_{\beta}(s)}{(\omega_{\alpha}^2 - \omega^2)\,(\omega_{\beta}^2 - \omega^2)} \tag{4.5}$$

where $$F_{\alpha\beta} = \int_s \bar{n}_u \times \bar{F}_{\alpha}(u) \cdot \overleftrightarrow{\mathcal{S}} \cdot \bar{F}_{\beta}(u)\, Z_s\, dS(u) =$$

$$= \frac{(1-j)\sqrt{\omega_{\alpha}\omega_{\beta}}}{Q_{\alpha\beta}} \tag{4.6}$$

where $$Q_{\alpha\beta} = \frac{\sqrt{\left(\omega_{\alpha}\int_V \frac{\mu F_{\alpha}^2}{2} dV\right)\left(\omega_{\beta}\int_V \frac{\mu F_{\beta}^2}{2} dV\right)}}{\frac{1}{2}\int_s \sqrt{\frac{\omega\mu}{2\sigma}}\, \bar{F}_{\alpha}(r) \cdot \bar{F}_{\beta}(r)\, dS(r)} \tag{4.7*}$$

The minus sign for F in equation (4.6) comes from the boundary condition (1.11) and the definition of the dyadic in terms of $\bar{H}$ and the tangential $\bar{E}$:

$$\bar{E}_t = (\bar{n} \times \bar{H})\, Z_s = -\bar{H} \cdot \quad Z_s$$

The Q is the internal Q due to the wall losses. If there are other irises or coupling devices, Q is the loaded Q including the wall losses and the energy lost through all coupling devices except the input iris.

Equation (4.5) is the first approximation:

Putting equation (4.5) in another form gives:

$$G_{11}^{(2)} = j\omega \sum_{\alpha} \frac{\bar{F}_{\alpha}(x)\bar{F}_{\alpha}(s)}{(\omega_{\alpha}^2 - \omega^2)} + \omega^2 \sum_{\alpha}\sum_{\beta} F_{\alpha\beta} \frac{\bar{F}(x)\bar{F}(s)}{(\omega_{\alpha}^2 - \omega^2)(\omega_{\beta}^2 - \omega^2)} \tag{4.8}$$

Substituting equation (4.8) under the integral sign of equation (1.12) then gives a second approximation: , and substituting again gives a third approximation:

* Note that this definition of Q differs from that in Montgomery, Principles of Microwave Circuits (1948) p 222, in that the $\sqrt{\omega_{\alpha}\omega_{\beta}}$ in the wall impedance is replaced by ω

The third approximation is:

$$\overleftrightarrow{G}_3^{(2)}(x,s) = -j\omega \left\{ \begin{aligned} &\sum_\alpha \frac{\bar{F}_\alpha(x)\bar{F}_\alpha(s)}{(\omega_\alpha^2-\omega^2)} \\ &-(j\omega)^1 \sum_\alpha \sum_\beta F_{\alpha\beta} \frac{\bar{F}_\alpha(x)\bar{F}_\beta(s)}{(\omega_\alpha^2-\omega^2)(\omega_\beta^2-\omega^2)} \\ &-(j\omega)^2 \sum_\alpha \sum_\beta \sum_\gamma \frac{F_{\alpha\beta}F_{\beta\gamma}\bar{F}_\alpha(x)\bar{F}_\gamma(s)}{(\omega_\alpha^2-\omega^2)(\omega_\beta^2-\omega^2)(\omega_\gamma^2-\omega^2)} \\ &-(j\omega)^3 \sum_\alpha \sum_\beta \sum_\gamma \sum_\delta \frac{F_{\alpha\beta}F_{\beta\gamma}F_{\gamma\delta}F_\alpha(x)F_\delta(s)}{(\omega_\alpha^2-\omega^2)(\omega_\beta^2-\omega^2)(\omega_\gamma^2-\omega^2)(\omega_\delta^2-\omega^2)} \end{aligned} \right\} \tag{4.9}$$

The nth approximation can be whitten down by inspection of equation (4.9). To be of practical use in calculations, the perturbed Green's function must be obtained in closed form (i.e. the function must be in closed form at each resonance).

4.2 Perturbed Green's Function.

To find a simple closed form for the basic terms in the perturbed Green's function of equation (4.10), assume that the cross coupling terms can be neglected so that $\overleftrightarrow{G}_n^{(2)}(x,s)$ becomes from the diogonal terms of equation (4.9), extended to the nth approximation:

$$\overleftrightarrow{G}_n^{(2)}(x,s) = +j\omega \sum_\alpha \bar{F}(x)\bar{F}(s)\, \Psi_\alpha(\omega_\alpha, F_{\alpha\alpha}) \tag{4.11}$$

$$\Psi_\alpha = \cancel{\frac{1}{(\omega_\alpha^2-\omega^2)}} + \sum_{k=0}^{n} \frac{(j\omega F_{\alpha\alpha})^k}{(\omega_\alpha^2-\omega^2)^{k+1}} \tag{4.12}$$

To have a finite admittance at resonance

Ψ_α should converge to something like the following:

$$\Psi_\alpha = \frac{1}{\omega_\alpha^2 - \omega^2 + j\omega F_{\alpha\alpha}}$$

Expanding Ψ_α and Φ_α and comparing terms:

Let $t = \frac{\omega_\alpha}{\omega}$, then by Peirce 748:

$$\Psi_\alpha = -\frac{1}{\omega^2}\left\{ \begin{array}{l} 1 \left[1 + t^2 + t^4 + t^6 + \cdots\cdots \right] \\ - \dfrac{j\omega F_{\alpha\alpha}}{\omega^2}\left[1 + 2t^2 + 3t^4 + 4t^6 + \cdots\cdots \right] \\ + \dfrac{(j\omega F_{\alpha\alpha})^2}{\omega^4}\left[1 + 3t^2 + 6t^4 + 10t^6 + \cdots \right] \\ - \dfrac{(j\omega F)^3}{\omega^6}\left[1 + 4t^2 + 10t^4 + 20t^6 + \cdots \right] \\ + \cdots\cdots\cdots \end{array} \right\} \tag{4.13}$$

Expanding Φ_{α} by Peirce 746:

$$= \frac{1}{(\omega_{\alpha}^2 - j\omega F_{\alpha\alpha}) - \omega^2}$$

$$= -\frac{1}{\omega^2}\left[1 + \left(\frac{\omega_{\alpha}^2 + j\omega F_{\alpha\alpha}}{\omega^2}\right) + \left(\frac{\omega_{\alpha}^2 - j\omega F_{\alpha\alpha}}{\omega^2}\right)^2 + \left(\frac{\omega_{\alpha}^2 - j\omega F_{\alpha\alpha}}{\omega^2}\right)^3 + \cdots\cdots\right] =$$

$$= -\frac{1}{\omega^2}\left\{\begin{array}{l} 1\left[1 + t^2 + t^4 + t^6 + \cdots\cdots\cdots\cdots\cdots\cdots\right] \\ -\left(\dfrac{j\omega F_{\alpha\alpha}}{\omega^2}\right)\left[1 + 2t^2 + 3t^4 + \cdots\cdots\right] \\ -\left(\dfrac{j\omega F_{\alpha\alpha}}{\omega^2}\right)^2\left[1 + 3t^2 + \cdots\cdots\cdots\cdots\right] \\ -\left(\dfrac{j\omega F_{\alpha\alpha}}{\omega^2}\right)^3\left[1 + \cdots\cdots\cdots\cdots\cdots\cdots\right] \end{array}\right\} \qquad (4.14)$$

Comparing equation (4.13) and (4.14) shows that they correspond term for term. Although the individual rows of these equation diverge for $\omega_{\alpha} \geq \omega$ the function Φ_{α} has the finite value $-1/jwF_{\alpha\alpha}$ at resonance. Since Ψ_{α} has been shown to correspond term for term with a series which converges, Ψ_{α} also converges.

The magnetic field in the iris contains the following components:

$$H_{iris} = H^{o}_{incident} + H^{o}_{reflected} + H'_{reflected} + H_{absorbed} \tag{5.3}$$

Defining: $I_i V_i = \int_s \bar{h}_i \cdot (\bar{n}_i \times \bar{E}_i)\, dS$ (5.4)

proprotional to the power into the region,

and noting $\bar{H}_i = I_i h_i$ then equation (1.8) gives:

$$V_i = \int_{iris} h_i \cdot (n_i \times E_i)\, dS = \int_{iris} h_i \cdot (n_i \times E)\, dS \tag{5.5}$$

The replacement of E_i by E in equation (5.5) is permitted by the orthogorality properties of the normal mode vectors.

Letting $I^o = I^o_{inc} - I^o_{refl}$ and substituting equations (5.3) and (5.5) into (5.2) gives:

$$I^o V^o_B = \int_{iris} H'_{refl} \cdot (n_B \times E)\, dS + \int_{iris} H_{abs} \cdot (\bar{n}_A \times E)\, dS \tag{5.6}$$

Substituting the integral of equation (1.5) in place of $\bar{H}'_{refl}$ and H_{abs} in equation (5.6) gives:

$$I^o V^o_B = + \left\{ \begin{array}{l} \int dS_r \int dS_s \left[\bar{n}_B \times \bar{E}(r)\right] \cdot \left[\overleftrightarrow{\Gamma}_B^{(2)'}(r,s) \cdot \left[\bar{n}_B \times \bar{E}(s)\right]\right] \\ + \int dS_r \int dS_s \left[\bar{n}_A \times \bar{E}(r)\right] \cdot \left[\overleftrightarrow{\Gamma}_A^{(2)}(r,s) \cdot \left[\bar{n}_A \times \bar{E}(s)\right]\right] \end{array} \right\} \tag{5.7}$$

A prime on a Green's function indicates that the dominant mode term is omitted.

The relative admittance which would be observed by measurement of the standing wave ratio and the position of the voltage minimum at a point to the left of plane 2-2' is:

$$\frac{Y}{Y_o} = \frac{I^o}{Y_o V^o_B} = \frac{I^o\ V^o_B}{Y_o (V^o_B)^2} \tag{5.8}$$

The characteristic admittance Y_o of the waveguide is defined to be consistent with Y_i in $\Gamma_B^{(2)}$ as defined in Appendix B.

Combing equations (5.7) and (5.8) and using the following principal mode voltages:

$$V_B^o = \int_{iris} n_B \times E(r) \cdot h_{oB} dS(r) \qquad \text{(waveguide)}$$

$$V_A^o = \int_{iris} \bar{n}_A \times \bar{E}(r) \cdot \bar{F}_{oA} dS(r) \qquad \text{(resonator)} \tag{5.9}$$

gives the following integral equation:

$$\frac{Y}{Y_o} = \frac{1}{Y_{1,0}} \left\{ \frac{\int\int dS_r\, dS_s\, \bar{n}_B \times \bar{E}(r) \cdot \Gamma_B^{(2)'}(r,s) \cdot \bar{n}_B \times \bar{E}(s)}{(V_A^o)^2} - \left(\frac{V_A^o}{V_B^o}\right)^2 \left[\frac{\int\int dS_r\, dS_s\, \bar{n}_A \times \bar{E}(r) \cdot \Gamma_A^{(2)'}(r,s) \cdot \bar{n}_A \times \bar{E}(s)}{\quad} + \frac{j\omega}{\omega_o^2 - \omega^2 + j\frac{\omega\omega_o}{Q_o}} \right] \right\} \tag{5.10}$$

Equation (5.10) is in stationary form like equation (1.10). Therefore, a good trial value of $E(r)$ will give a close approximation to Y/Y_o.

5.2 Equivalent Circuit

Inspection of equation (5.10) shows it to be in the form:

$$\frac{Y}{Y_o} = \frac{1}{Y_o} \left\{ Y_B' + N^2 \left(Y_A' - Y_{AO} \right) \right\} \tag{5.11}$$

This form leads to the following equivalent circuit:

The transformer ratio can be a complex number on account of the curvative of the wall separating the waveguide and the resonator. To take account of the finite wall thickness of the iris, the ideal transformer can be replaced by two ideal transformers with a short section of waveguide (transmission line) between them. When the iris width is less than a half wavelength the two transformer ratios would be complex due to the imaginary nature of the admittance of a waveguide at a frequency below the cutoff frequency. For most practical cases the approximate correction for wall thickness of section 3.2 is reasonable.

5.3 Shift in Resonant Frequency and the Coupled Q

At resonance, Y is real, so the resonant frequency can be determined by plotting Y_{AO}, Y'_A, and Y'_B as functions of frequency and graphically finding the frequency for which:

$$\mathrm{Im}\ Y = j B = 0 \qquad (5.12)$$

The coupled Q can then be found from the slope of the susceptance curve at the resonant frequency[24]:

$$Q_c = \frac{\ }{2Y_o} \frac{B}{\ } \qquad (5.13)$$

24. Montgomery, C.G. ~~Microwave/Networks~~ Principles of Microwave Circuits (1948). p. 230.

To make the transformer ratio N equal to one for simplification, the waveguide is made to have the same dimensions as the resonator. This reduces equation (6.2) to:

$$\frac{Y}{Y_o} = \frac{1}{Y_o}\left[\, Y_B' + Y_A' + Y_{Ao} \right] \tag{6.27}$$

Using the approximation of equation (6.26) for all modes except the $TE_{1,0}$ mode and using equations (6.8) and (6.12) to simplify (6.27) results in:

$$\frac{Y}{Y_o} = \frac{1}{Y_o}\left[\, G_{1,0,1} + j\left(B_m - \cot \kappa_{1,0} L\right)\right] \tag{6.28}$$

The resonant frequency is determined by the value of $_{1,0}$ which makes the susceptance zero as required by equation (5.12). The iris susceptance B_m/Y_o is taken from the curves of Marcuvitz, items 1 and 6 in Appendix D. The resultant resonant frequencies for the resonator of figure 6.2 are plotted in figure 6.3 in the form of the fractional shift in resonant frequency from the frequency of the unloaded resonator.

Part of the range of the $TE_{1,0,2}$ mode is shown in figure 6.3 to illustrate the behavior of capacitive irises. Examination of the curves shows that if an capacitive iris of zero height (= 0) and full waveguide width ($\alpha = 1.0$) were cut in a $TE_{1,0,1}$ resonator, there would be no frequency shift, but as the height is increased or the width decreased, the frequency would increase along curves in figure 6.3 which join up with the next higher mode, the $TE_{1,0,2}$ mode. Thus starting with a thin capacitive iris in the $TE_{1,0,1}$ mode, then increasing the height until it is a resonant iris, followed by decreasing the width through the inductive region one ends up width the $TE_{1,0,2}$ mode.

This transition between two modes as the iris dimensions are changed has its counterpart in accoustical theory. Harris and Feshbach have shown that the accoustical (1,0,0) mode shifts in resonant frequency toward the (2,0,0) modeas the door between two rectangular rooms is widened.[29]

To obtain the coupled Q, equation (5.13) is used in a manner similar to that given by Montgomery[30] in which the energy stored in the resonator and in the waveguide at the iris is obtained by use of the derivative of the susceptance at a pole of the reactance. This formula applies strictly to losseles

$$Q_c = \frac{\omega}{2 Y_o} \frac{\partial B}{\partial \omega} = \frac{\pi \lambda_g L}{\lambda^2} \left[1 + \left(\frac{B_m}{Y_o} \right)^2 \right] \qquad (6.29)$$

networks, but is a good approximation for highinternal Q's as are usually obtained in microwave cavity resonators. Since the investigation of large irises covers a range of λ_g between 2L and 4L the conditions $L \simeq \lambda_g/2$ and $(B/Y_o)^2 >> 1$ used by Montgomery cannot be used in this case. These two conditions are not contained in equation (6.29), so this equation can be used for the full range of iris dimensions. Curves of Q_c as a function of iris dimensions for the rectangular resonator of figure 6.2 are plotted in figure 6.4. These curves are obtained by finding the $\Delta\omega/\omega$ for a given α and β from figure 6.3 and the corresponding B_m/Y_o from the curves of item 6 of Appendix D,

29. Harris, C. M. and Feshbach, H. Jour. Accoustical Soc. of Amer. Vol. 22, Sept 1950. pp. 572-578.

30. Montgomery, C. G. Principles of Microwave Circuits (1948), pp. 230-234.

and then inserting , $_g$, and B_m/I_o into equation (6.29).

6.4 Use of Rectangular Resonator Results as an Approximation for Resonators of Other Shapes.

The curves of $\Delta\omega/\omega$ and Q_c as a function of iris dimensions can be converted into the functions v and q by use of equations (3.4), (3.5), (3.3). This permits the results on this particular $TE_{1,0,1}$ rectangular resonator to be used as an approximation for other resonators having a similar field distribution in the iris. This approximation is equivalent to using the Green's functions for this rectangular resonator in place of the correct Green's functions of the resonator under investigation. in addition to the other approximations already made in this rectangular resonator analysis.

Using the energy storage formulas from Ramo and Whinnery[31]:

$$U_E = \epsilon \frac{abL}{8} E_o^2 \qquad H = E_o\sqrt{\frac{\epsilon}{\mu}}\frac{\lambda}{\lambda_g}$$

in Equation (3.3) gives:

31 Ramo, S. and Whinnery, J. R. Fields and Waves in Modern Radio (1944). pp. 383-389.

which would be more likely to produce closer agreement with experiment is to use a more accurate trial field in the iris.

For $\beta = 0.1$ the approximate curve A departs more severely from the experimental results for irises larger than $\alpha = 0.6$. To obtain a more accurate curve for larger widths, curve B was calculated by use of equation (7.32) in which the $TE_{2,1,1}$ mode which crosses the principal mode is simply omitted. The frequencies ~~used~~ used in this calculation were the approximate theoretical frequencies of figure 7.2. This approximation comes closer to the shape of the experimental curve. To obtain greater accuracy it would probably be necessary to both include a second term in the trial field in the iris and to retain the wall loss terms in the Green's functions for the principal mode, the crossing mode, and possibly the closest modes on each side of the reonant frequency. When the wall losses are included, the simple formula (5.13) for the coupled Q is no longer valid, so for each iris size a number of points near the resonant frequency have to be calculated, so that the calculations become prohibitive. For practical calculations the rectangular approximation of the experimental results on a similar cavity can be used to obtain a reasonable approximation for the coupled Q.

A [illegible] calculation [illegible]

susceptance at resonance. For large power output the iris susceptance contributes ~~very little~~ less to the ~~resonator~~ total susceptance so that the resonant frequency shifts to a value where the resonator appears more like a quarter wave shorted line than the original half wave line.

The extension of Bethe's formulas by replacing the lumped constants by constants determined by the susceptance in an infinite waveguide doesn't extend the analysis to the large power output case, because it doesn't take into account the zero iris susceptance condition. However the transverse iris susceptance in a waveguide can be used to obtain reasonable approximate results, if the resonator has fields at the iris which approximate some resonator with axes of symmetry which permit it to be treated as a shorted length of transmission line.

The integral equation for the input admittance of a cavity resonator coupled to a waveguide through an iris when put in J. S. Schwinger's varitional form, yields a relatively simple equivalent circuit. However at the junction of a curved wall resonactr with a rectangular waveguide the explicit formulas for the susceptance of waveguide modes come out in a series form which contains the sum and difference of a great number of terms of the same order of magnitude, which makes precise calculations impractical in such cases.

There is a difference in the way the waveguide and resonator modes contribute to the susceptance in the equivalent circuit. The modes on the waveguide side appear as inductance for the TE modes and capacitance for the TM modes. On the resonator side the mdes whose resonant frequencies are below the operating frequency appear as capacitances while those above appear as inductances. This difference is due to the situation where the axis of the waveguide and the axis of the resonator are not in the same direction.

Close agreement has been obtained between approximate theory and experiment for the coupled Q of a coaxial resonator coupled to the a rectangular waveguide from the the curved side of the resonator through an inductive iris. The fact that close agreement was obtained with the following approximations warrants careful consideration:

(1) The curvature of the iris is neglected.

(2) The Green's functions for a rectangular resonator are used instead of those for a coaxial resonator.

(3) The position of the iris is shifted from the second maximum of the magnetid field to the first maximum at the enc of the resonator.